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NUMERICAL ANALYSIS R V KOHN ET AL. OCT 84 BN-1028

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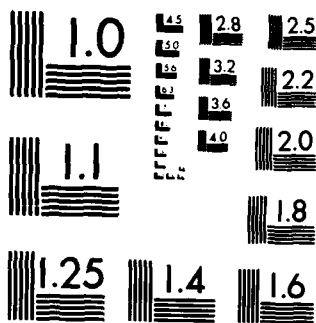
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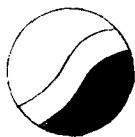
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Determining conductivity by boundary measurements II.

Interior results.

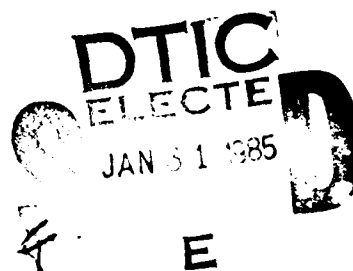
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Determining conductivity by

boundary measurements II.

Interior results.

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Abstract

In a recent paper [6] ^{the authors} we showed that an unknown real-analytic conductivity γ ^{can be} may be determined from static boundary measurements. ^{In this paper they} Here we extend this analysis by demonstrating that a similar result holds for piecewise real-analytic conductivities. In addition, for the special case of a layered structure ^{it is shown} ~~we show~~ that a three times continuously differentiable conductivity is identifiable by boundary measurements. *Additional results: uniqueness, convergence, algorithms, estimates, coefficients.*

1. Introduction

This paper addresses the following inverse problem: Can one determine an unknown conductivity γ inside a body Ω by means of static measurements at the boundary? Since there is no dependence on time, the underlying equation $\nabla \cdot (\gamma \nabla u) = 0$ is elliptic. Our first paper [6] established the identifiability of a real-analytic γ ; in this one we demonstrate a similar result when γ is piecewise real-analytic (for example, piecewise constant).

Consider a bounded domain $\Omega \in \mathbb{R}^n$, $n \geq 2$, and a "conductivity" $\gamma \in L^\infty(\Omega)$, $0 < \gamma_0 \leq \gamma(\underline{x})$. For appropriate ϕ there is a unique $u \in H^1(\Omega)$ such that

$$(1) \quad \nabla \cdot (\gamma(\underline{x}) \nabla u) = 0 \quad \text{in } \Omega$$

$$u = \phi \quad \text{on } \partial\Omega,$$

obtained for example by minimizing Dirichlet's integral. Let $Q_Y(\phi)$ be the energy of the solution,

$$Q_Y(\phi) = \int_{\Omega} \gamma |\nabla u|^2 d\underline{x}.$$

If $\partial\Omega$ is sufficiently smooth then Green's formula gives

$$Q_Y(\phi) = \int_{\partial\Omega} u \cdot \gamma \frac{\partial u}{\partial \nu} ds,$$

where "ds" denotes surface area; consequently $Q_Y(\phi)$ depends only on the Dirichlet-data ϕ and its associated Neumann-data, the conormal derivative $\gamma \frac{\partial u}{\partial \nu}$. Our inverse problem is to determine γ given knowledge of the quadratic form Q_Y . We shall say that γ is identifiable (within a certain class) by boundary measurements if the map $\gamma \mapsto Q_Y$ is injective (in this class).

There is an equivalent formulation in terms of the map taking Dirichlet- to Neumann-data

$$\Lambda_\gamma : u|_{\partial\Omega} \rightarrow \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

The form Q_γ determines the map Λ_γ , by polarization. Therefore our problem is alternately to determine γ given knowledge of the Dirichlet- to Neumann-data map Λ_γ .

If γ and $\frac{\partial \gamma}{\partial \nu}$ are known at $\partial\Omega$, then there is another formulation involving $-\Delta + q$. Indeed, $v = \gamma^{1/2}u$ solves

$$-\Delta v + qv = 0 \quad \text{in } \Omega$$

$$v = \gamma^{1/2}\phi \quad \text{on } \partial\Omega,$$

with

$$(2) \quad q = -\frac{1}{4} \frac{|\nabla \gamma|^2}{\gamma^2} + \frac{1}{2} \frac{\Delta \gamma}{\gamma}$$

whenever u solves (1), assuming sufficient regularity of γ . The Dirichlet- to Neumann-data map for q is

$$\tilde{\Lambda}_q : v|_{\partial\Omega} \rightarrow \frac{\partial v}{\partial \nu}|_{\partial\Omega}.$$

If the boundary values of γ and $\frac{\partial \gamma}{\partial \nu}$ are known then information about $\tilde{\Lambda}_q$ is equivalent to that about Λ_γ ; hence our problem is also to find the unknown potential $q(x)$ given knowledge of the map $\tilde{\Lambda}_q$.

Some special cases of the inverse problem can be treated by separation of variables. R. E. Langer considered a layered half-space in 1933 [8]. He reconstructed all the derivatives of γ at the boundary from knowledge of $\Lambda_\gamma^{-1}(\psi_0)$ for just one particular flux ψ_0 . Moreover his choice of ψ_0 is one which naturally

shows up in geophysical applications. Another example is that of a layered rectangle, analyzed by Cannon, Douglas and Jones in 1963 [2].

The general case, without restrictions on the form of Ω and γ , has only been considered more recently; to the best of our knowledge it was first raised by A. P. Calderón [1]. He proved that the map $\gamma \rightarrow Q_\gamma$ is Frechet differentiable for $\gamma \in L^\infty$, and that the differential at $\gamma = \text{constant}$ is injective. However its range is not closed, so the implicit function theorem does not apply, and one cannot conclude the identifiability of γ from this analysis.

We took a different approach in [6], using the variational principle and special Dirichlet data with localized, highly oscillatory behavior. We proved that the quadratic form Q_γ (or the map Λ_γ) determines all the derivatives of γ at the boundary. Identifiability in the class of real-analytic γ follows as an immediate corollary. A more extensive review of the literature is found in [7]. It seems worth noting that the method of [6] can also be used for $-\Delta + q$, even when q does not have the form (2). If q and $\partial\Omega$ are C^∞ then the Dirichlet-to Neumann-data map $\tilde{\Lambda}_q$ or the corresponding energy form $q \rightarrow \int (|\nabla v|^2 + q|v|^2) d\mathbf{x}$ (provided each is well-defined) determines q and all its derivatives at the boundary.

Unfortunately, the results just summarized are far from satisfactory. The original goal was to reconstruct γ everywhere, while these results give only its derivatives at the boundary. It remains unknown, for example, whether a C^∞ conductivity is determined in the interior by these boundary measurements.

The present paper represents some modest progress toward the interior reconstruction problem. Our main result is one of identifiability in the class of piecewise real-analytic conductivities. The proof, presented only in dimension two for ease of exposition, uses the ideas of [6] together with the Runge

Approximation Property for solutions of $\nabla \cdot (\gamma \nabla u) = 0$. Its concept is sketched in section 2, and the details are developed in sections 3-5. Our second result, presented in section 6, is the identifiability of a layered conductivity which is merely three times continuously differentiable. For simplicity we treat only the case of a finite-width strip in \mathbb{R}^2 . The proof uses separation of variables and recent progress in one-dimensional inverse spectral theory. Our third topic, presented in section 7, is the convergence of a reconstruction algorithm for real-analytic γ . This algorithm restricts its attention to approximations of γ within a suitable compact set and the proof of convergence depends heavily on the identifiability demonstrated in [6]. The proof gives no information on the rate of convergence or the efficiency of the algorithm; these remain important directions for further study.

§2. Concept of the piecewise analytic case

We begin by reviewing how boundary measurements were shown to determine a real-analytic conductivity in [6]. Given a smoothly bounded domain Ω , consider two coefficients γ_1 and γ_2 , and let u_1 and u_2 be the corresponding solutions of (1):

$$(3) \quad \nabla \cdot (\gamma_i \nabla u_i) = 0, \quad u_i|_{\partial\Omega} = \phi, \quad i = 1, 2.$$

If $\gamma_1 \neq \gamma_2$ then their Taylor expansions differ; relabeling if necessary, one easily concludes that

$$\gamma_1(\underline{x}) - \gamma_2(\underline{x}) \geq C \rho(\underline{x})^\ell, \quad \rho(\underline{x}) = \text{dist}(\underline{x}, \partial\Omega)$$

for some $\ell \geq 0$, in a $\bar{\Omega}$ -neighborhood D of some $\underline{x}_0 \in \partial\Omega$. By choosing Dirichlet data ϕ that oscillate rapidly near \underline{x}_0 , one can arrange that the energy of u_1 be concentrated in D , and indeed that

$$(4) \quad \int_{\Omega \setminus D} |\nabla u_1|^2 d\underline{x} < \varepsilon \int_D \rho^\ell |\nabla u_1|^2 d\underline{x}$$

with $\varepsilon > 0$ as small as desired (see Lemma 3). Then

$$\begin{aligned} (5) \quad Q_{\gamma_1}(\phi) &= \int_{\Omega} \gamma_1 |\nabla u_1|^2 d\underline{x} \geq \int_D \gamma_1 |\nabla u_1|^2 d\underline{x} \\ &\geq \int_D \gamma_2 |\nabla u_1|^2 d\underline{x} + C \int_D \rho^\ell |\nabla u_1|^2 d\underline{x} \\ &> \int_{\Omega} \gamma_2 |\nabla u_1|^2 d\underline{x} \\ (6) \quad &\geq \int_{\Omega} \gamma_2 |\nabla u_2|^2 d\underline{x} = Q_{\gamma_2}(\phi), \end{aligned}$$

using (4) and the variational principle for (3) in the last two steps. Therefore

for this choice of ϕ the boundary measurements are different.

Our method for the piecewise analytic case is a direct extension of this. Consider the simple example of the ball $\Omega = \{|\underline{x}| < 1\}$ with an unknown concentric inclusion:

$$\gamma_i(\underline{x}) = \begin{cases} \gamma_i'(\underline{x}) & |\underline{x}| < r_i \\ \gamma_i''(\underline{x}) & |\underline{x}| > r_i \end{cases} \quad i = 1, 2.$$

Here γ_i' and γ_i'' are real-analytic functions of \underline{x} . If γ_1 and γ_2 have the same boundary measurements then the argument sketched above shows that $\gamma_1'' = \gamma_2''$ in $|\underline{x}| > \max(r_1, r_2)$. A new idea is required, however, to establish that $\gamma_1 = \gamma_2$ also for $|\underline{x}| < \max(r_1, r_2)$.

Assuming that $r_1 \leq r_2$, what we would really like to do is to repeat the estimation

$$\int_{\Omega} \gamma_1 |\nabla u_1|^2 d\underline{x} > \int_{\Omega} \gamma_2 |\nabla u_1|^2 d\underline{x}$$

with Ω replaced by the subdomain $\Omega' = \{|\underline{x}| < r_2\}$. At first glance this seems impossible, since the elliptic equation $\nabla \cdot (\gamma \nabla u) = 0$ will smooth out any oscillations of the Dirichlet data prescribed on the outer boundary. But in fact it is possible, since this equation has the Runge Approximation Property: if $\nabla \cdot (\gamma \nabla u) = 0$ on a subdomain $\omega \subset \Omega$, then u can be approximated on compact subsets of ω by solutions of the same equation in the full domain Ω (see Lemma 2).

Therefore to show that $\gamma_1 = \gamma_2$ across $|\underline{x}| = r_2$, we once again assume the contrary: if not, then

$$|\gamma_1(\underline{x}) - \gamma_2(\underline{x})| \geq C[\rho'(\underline{x})]^\ell, \quad \rho'(\underline{x}) = r_2 - |\underline{x}|,$$

for some $\ell \geq 0$, in a neighborhood D' relative to $\bar{\Omega}' = \{|\underline{x}| \leq r_2\}$ of some $\underline{x}_0 \in \partial\Omega'$. We suppose for simplicity that γ_1 is the larger in this neighborhood. The construction of [6] together with the Runge property leads to a function u_1 with $\nabla \cdot (\gamma_1 \nabla u_1) = 0$ on all of Ω and

$$\int_{\Omega \setminus D'} |\nabla u_1|^2 d\underline{x} < \varepsilon \int_{D'} (\varepsilon')^\ell |\nabla u_1|^2 d\underline{x}$$

(see Lemma 4). The behaviour of u_1 off Ω' is essentially unknown, but that is acceptable because $\gamma_1 = \gamma_2$ there. Estimates parallel to (5)-(6) using $\phi = u_1|_{\partial\Omega}$ show that $Q_{\gamma_1}(\phi) > Q_{\gamma_2}(\phi)$ when ε is small enough. Therefore if γ_1 and γ_2 had the same boundary data they must have been equal in a neighborhood of $|\underline{x}| = r_2$.

If $r_1 = r_2$ this argument establishes $\gamma_1 = \gamma_2$. If $r_1 < r_2$ then we've shown that $\gamma_1 = \gamma_2$ in $\{|\underline{x}| > r_1\}$; the same argument can be repeated at $|\underline{x}| = r_1$ to establish $\gamma_1' = \gamma_2'$ in $\{|\underline{x}| < r_1\}$, and it follows that $\gamma_1 = \gamma_2$. Though the geometrical and analytical technicalities are substantial, this procedure of "marching inward from the boundary" will work in essentially the same way for the general piecewise analytic case.

3. Preliminaries

We begin with definitions and conventions on the use of the term "piecewise analytic". From this point on we drop the prefix real and use the term analytic as a synonym for real-analytic. Let ω be an open set in \mathbb{R}^m , $m = 1$ or 2 . We recall that a function $\gamma : \omega \rightarrow \mathbb{R}$ is analytic in ω if it is infinitely often differentiable and for every compact subset $K \subset \omega$ there exists a constant r_K for which

$$\sup \frac{1}{\alpha!} r_K^{|\alpha|} |D^\alpha \gamma(\underline{x})| < \infty,$$

the supremum being over all points $\underline{x} \in K$ and all m -tuples α of nonnegative integers.

We shall say that a C^∞ function γ is analytic on $\bar{\omega}$ if it has an extension which is analytic in a neighborhood of $\bar{\omega}$. A mapping $g : [0,1] \rightarrow \mathbb{R}^2$ is called an analytic curve if each of its coordinates is analytic on $[0,1]$ and $Dg(s) \neq 0$ for all $s \in [0,1]$.

A bounded, open set $\omega \subset \mathbb{R}^2$ is called a piecewise analytic domain if

- (i) ω is connected,
- (ii) its boundary $\partial\omega$ is a union of finitely many (images of) analytic curves,
- (iii) ω lies locally on one side of $\partial\omega$.

Notice that the boundary of a piecewise analytic domain can have cusps.

Following [3] we say that a bounded, open set $\omega \subset \mathbb{R}^2$ is an analytic curvilinear polygon if it is connected, and if for each $\underline{x} \in \partial\omega$ there exists a neighborhood B of \underline{x} (relative to \mathbb{R}^2) and a map \dagger such that

- (i) ϕ maps B injectively onto a neighborhood of the origin in \mathbb{R}^2
- (ii) ϕ and ϕ^{-1} have analytic coordinate functions
- (iii) $\phi(\omega \cap B)$ is either $\{\underline{x} : x_2 > 0\} \cap \phi(B)$, $\{\underline{x} : x_1 > 0 \text{ and } x_2 > 0\} \cap \phi(B)$ or $\{\underline{x} : x_1 > 0 \text{ or } x_2 > 0\} \cap \phi(B)$.

Analytic curvilinear polygons are special cases of piecewise analytic domains without cusps and "corners" of angle π .

A family $\{\omega_j\}_{j=1}^N$ is a (disjoint) piecewise analytic cover of a closed set $\bar{\omega}$ if

- (i) each ω_j is a piecewise analytic domain,
- (ii) $\bar{\omega} \subseteq \bigcup_{j=1}^N \bar{\omega}_j$
- (iii) $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$.

A function $\gamma : \omega \rightarrow \mathbb{R}$ is piecewise analytic on $\bar{\omega}$ (relative to the cover $\{\omega_j\}$) if it is analytic on each $\bar{\omega}_j$, $1 \leq j \leq N$. Notice that no continuity is assumed across the interfaces.

In proving identifiability we shall consider two conductivities γ_1 and γ_2 corresponding to different piecewise analytic covers. It is important to note that they are both piecewise analytic relative to a common refinement of the two covers:

Lemma 1. If γ_1 and γ_2 are piecewise analytic functions on $\bar{\omega}$, then there exists a piecewise analytic cover $\{\omega_j\}_{j=1}^N$ such that both γ_1 and γ_2 are analytic on each $\bar{\omega}_j$, $1 \leq j \leq N$.

Proof: Let $\{\omega_j^{(i)}\}_{j=1}^N$ be a cover relative to which γ_i is piecewise analytic, $i = 1, 2$. For any pair (j, k) the boundary of the intersection $\omega_j^{(1)} \cap \omega_k^{(2)}$

consists of finitely many (images of) analytic curves. Of course $\omega_j^{(1)} \cap \omega_k^{(2)}$ may not be connected; also, as shown in figure 1, the boundary might intersect

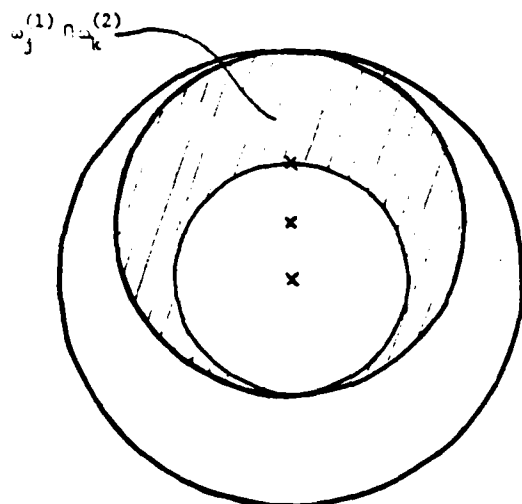


Fig. 1

$$\omega_j^{(1)} = \{\underline{x} \in \mathbb{R}^2 : 1/2 < |\underline{x}| < 1\}$$

$$\omega_k^{(2)} = \{\underline{x} \in \mathbb{R}^2 : |\underline{x} - (0, 1/4)| < 3/4\}$$

itself at isolated points. Nevertheless, by considering the connected components and "cutting out" a neighborhood of any point of self-intersection and dividing this neighborhood in separate pieces, we can obtain

$$\overline{\omega_j^{(1)} \cap \omega_k^{(2)}} = \bigcup_{\ell=1}^M \overline{\omega_\ell^*},$$

where the ω_ℓ^* are disjoint, piecewise analytic subdomains of $\omega_j^{(1)} \cap \omega_k^{(2)}$. The collection of all ω_ℓ^* (corresponding to all pairs (j,k)) constitutes a cover of Ω relative to which both γ_1 and γ_2 are piecewise analytic.

□

Remark 1.

In the preceding proof we used the fact that two analytic curves either intersect in at most finitely many points or the intersection is itself an analytic curve. This was used to conclude that the intersection of two piecewise analytic domains has a boundary which consists of finitely many analytic curves. A similar fact does not hold for C^k -curves, and this is why we work with piecewise

analytic portions of $\partial\Omega$. □

For an arbitrary open set $\omega \subseteq \mathbb{R}^m$, $m = 1, 2$, we use the terminology $H^k(\omega)$ to denote the set of functions that together with all their derivatives of order $\leq k$ are square integrable in ω . Spaces with noninteger indices are defined by complex interpolation. If g is an analytic curve with $\{g(s) : s \in (0, 1)\} = \Gamma$, then $H^t(\Gamma)$ denotes those functions u for which $u \circ g \in H^t((0, 1))$. The norm on $H^t(\omega)$ is denoted $\|\cdot\|_{t, \omega}$ and similarly for the space $H^t(\Gamma)$.

Let Ω be an analytic curvilinear polygon. We recall that $\underline{x}_0 \in \partial\Omega$ is a corner if near it Ω is analytically isomorphic to either $\{\underline{x} : x_1 > 0 \text{ and } x_2 > 0\}$ or $\{\underline{x} : x_1 > 0 \text{ or } x_2 > 0\}$. Let $\underline{x}(s)$, $s \in [-\delta, \delta]$, $\delta > 0$, be a parametrization of $\partial\Omega$ near $\underline{x}_0 = \underline{x}(0)$ according to arclength (i.e. the arclength from $\underline{x}(s)$ to \underline{x}_0 is $|s|$, and $\underline{x}(s)$ and $\underline{x}(-s)$ lie on opposite sides of \underline{x}_0). If ϕ is a function on $\partial\Omega$, we set

$$I_{\underline{x}_0}(\phi) = \int_0^\delta |\phi(\underline{x}(s)) - \phi(\underline{x}(-s))|^2 s^{-1} ds.$$

Let \underline{x}_i , $1 \leq i \leq M$, be all the corners of $\partial\Omega$, and let $\Gamma_i = \{g_i(s) : s \in (0, 1)\}$ where g_i , $1 \leq i \leq M$, are the analytic boundary curves connecting these corners. We define

$$\begin{aligned} \tilde{H}^{1/2}(\partial\Omega) = \{ \phi : \phi|_{\Gamma_i} \in H^{1/2}(\Gamma_i), \quad 1 \leq i \leq M \text{ and} \\ I_{\underline{x}_i}(\phi) < \infty, \quad 1 \leq i \leq M \}, \end{aligned}$$

with norm

$$\|\cdot\|_{\tilde{H}^{1/2}} = \left(\sum_{i=1}^M \left[\|\cdot\|_{\frac{1}{2}, \Gamma_i}^2 + I_{\underline{x}_i}(\cdot) \right] \right)^{1/2}.$$

There is a continuous, surjective trace operator from $H^1(\Omega)$ to $\tilde{H}^{1/2}(\partial\Omega)$ [3].

the statement " $u = \phi$ on $\partial\Omega$ " should always be interpreted in the sense of this trace.

For any $\gamma \in L^\infty$ with $0 < \gamma_0 \leq \gamma(\underline{x})$ we denote by L_γ the corresponding operator

$$L_\gamma u = \nabla \cdot (\gamma \nabla u) .$$

The equation

$$L_\gamma u = 0 \quad \text{in } \Omega , \quad u = \phi \quad \text{on } \partial\Omega$$

has a (unique) solution $u \in H^1(\Omega)$ exactly if $\phi \in \tilde{H}^{1/2}(\partial\Omega)$; the energy $\int_\Omega \gamma |\nabla u|^2 d\underline{x}$ is a continuous quadratic form on $\tilde{H}^{1/2}(\partial\Omega)$. Whenever $u \in H^1(\Omega)$ and $L_\gamma u \in L^2(\Omega)$, the conormal derivative

$$\gamma \frac{\partial u}{\partial \nu} \in [\tilde{H}^{1/2}(\partial\Omega)]^* \quad (\text{the dual of } \tilde{H}^{1/2})$$

is defined by

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} \cdot \nu ds = \int_\Omega \gamma \nabla u \cdot \nabla v d\underline{x} + \int_\Omega L_\gamma u \cdot v d\underline{x} ,$$

where $v \in H^1(\Omega)$ is any function satisfying

$$v = \psi \quad \text{on } \partial\Omega , \quad \|v\|_{1,\Omega} \leq C \|\psi\|_{\tilde{H}^{1/2}} .$$

The Dirichlet-to-Neumann operator $\Lambda_\gamma : u|_{\partial\Omega} \rightarrow \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ maps $\tilde{H}^{1/2}(\partial\Omega)$ to $[\tilde{H}^{1/2}(\partial\Omega)]^*$, and its image is the subspace annihilated by the constant functions.

We turn next to the Runge Approximation Property. Let ω be a subdomain of Ω and consider a solution of $L_\gamma u = 0$ in ω . It is generally not possible to extend u to a solution of $L_\gamma u = 0$ in the full domain Ω ; that would correspond to solving the Cauchy problem for this elliptic equation. It

is possible, however, to find an "approximate extension" in a certain sense. This follows from results of Lax [9] and Malgrange [11]; we give a complete proof for the reader's convenience.

Lemma 2. (The Runge Approximation Property) Let ω be a C^∞ domain contained in the analytic curvilinear polygon Ω , and such that each connected component of $\Omega \setminus \omega$ has a boundary curve in common with $\partial\Omega$. Let $0 < \gamma_0 \leq \gamma(x)$ be piecewise analytic on $\bar{\Omega}$ and assume that $u \in H^1(\omega)$ satisfies

$$L_\gamma u = 0 \text{ in } \omega.$$

Given any compact subset $K \subset \omega$ and any $\epsilon > 0$ there exists $U \in H^1(\Omega)$ such that

$$L_\gamma U = 0 \text{ in } \Omega, \text{ and}$$

$$\int_K |\nabla(U-u)|^2 d\underline{x} < \epsilon.$$

Proof: It will suffice to show that $\int_\omega |U-u|^2 d\underline{x}$ can be made as small as desired, since

$$\int_K |\nabla(U-u)|^2 d\underline{x} \leq C \int_\omega |U-u|^2 d\underline{x}$$

as a consequence of the identity $L_\gamma(U-u) = 0$ in ω . So we must show that

$$H = \{u : u=U|_\omega, U \in H^1(\Omega), L_\gamma U = 0 \text{ in } \Omega\}$$

is dense in

$$\{u : u \in L^2(\omega), L_\gamma u = 0 \text{ in } \omega\}$$

in the norm of $L^2(\omega)$.

If not then there must be some $u^* \in L^2(\omega) \setminus \{0\}$ satisfying

$$(7) \quad \begin{aligned} L_Y u^* &= 0 \quad \text{in } \omega \\ \int_{\omega} u^* v \, d\underline{x} &= 0 \quad \text{for all } v \in H. \end{aligned}$$

We shall derive a contradiction from this. Let $\phi \in H^1(\Omega)$ solve

$$(8) \quad L_Y \phi = U^* \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega,$$

where $U^* = u^*$ in ω , $U^* = 0$ in $\Omega \setminus \omega$. By (7)

$$\begin{aligned} 0 &= \int_{\omega} u^* v \, d\underline{x} = \int_{\Omega} L_Y \phi \cdot v \, d\underline{x} \\ &= \int_{\partial\Omega} \gamma \frac{\partial \phi}{\partial \nu} \cdot v \, ds \end{aligned}$$

for any $v \in H^1(\Omega)$ with $L_Y v = 0$ in Ω ; therefore $\gamma \frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$.

We recall from (8) that ϕ also vanishes at $\partial\Omega$. Since γ is piecewise analytic on $\bar{\Omega}$, $U^* = 0$ in $\Omega \setminus \omega$, and each component of $\Omega \setminus \omega$ has a boundary curve in common with $\partial\Omega$, it follows by repeated application of Holmgren's Uniqueness Theorem (see e.g. [4]) that

$$\phi = 0 \quad \text{in } \Omega \setminus \omega.$$

Therefore $\phi = \gamma \frac{\partial \phi}{\partial \nu} = 0$ on $\partial\omega$, and as a consequence

$$\int_{\omega} |u^*|^2 \, d\underline{x} = \int_{\omega} L_Y \phi \cdot u^* \, d\underline{x} = 0.$$

This contradicts the hypothesis that u^* was nonzero, and the proof is complete. □

Remark 2: It is not essential that ω be smoothly bounded or that Ω be an analytic curvilinear polygon. The hypothesis that γ be piecewise analytic can also be relaxed, by using Aronszajn's unique continuation theorem instead of Holmgren's. The version stated here, however, is exactly what will be needed in the proof of Lemma 4.



4. Energy estimates

The following is a slight reformulation of a result in [6].

Lemma 3. Let ω be a bounded C^∞ domain, x_0 a point on $\partial\omega$, and D a neighborhood of x_0 relative to $\bar{\omega}$. Assume that $\gamma \in L^\infty(\omega)$, $0 < \gamma_0 \leq \gamma(x)$, and that $\gamma \in C^\infty(D)$. Given $\ell \geq 0$ and $\varepsilon > 0$ there exists $u \in H^1(\omega)$ such that

$$(i) \quad L_\gamma u = 0 \quad \text{in } \omega$$

$$(ii) \quad \int_{\omega \setminus D} |\nabla u|^2 dx < \varepsilon \int_D \rho^\ell |\nabla u|^2 dx$$

with $\rho(x) = \text{dist}(x, \partial\omega)$.

Proof: For any fixed integer $M \geq 0$, Lemma 1 of [6] constructs a sequence

$\{\phi_N\}_{N=1}^\infty \subseteq C^\infty(\partial\omega)$ such that

$$(9a) \quad \|\phi_N\|_{\frac{1}{2}+t, \partial\omega} \leq C_t N^t, \quad t \geq -M,$$

$$(9b) \quad \|\phi_N\|_{\frac{1}{2}, \partial\omega} = 1$$

$$(9c) \quad \text{supp } \phi_N \rightarrow \{x_0\} \quad \text{as } N \rightarrow \infty.$$

We review the construction when x_0 is the origin and $\partial\omega = \{x : x_2 = 0\}$ (to which the general case is easily reduced). Let ψ be a C^∞ function on \mathbb{R} , not identically zero, with

$$\text{supp } \psi \subset [-1, 1],$$

$$\int_{-1}^{+1} s^k \psi(s) ds = 0 \quad \text{for integers } k, \quad 0 \leq k \leq M-1.$$

If $\psi_N(x_1) = \psi(Nx_1)$, one easily verifies that $\phi_N = \psi_N / \|\psi_N\|_{\frac{1}{2}, \partial\omega}$ has properties (9a-c).

Now consider the solution u_N of

$$L_Y u_N = 0 \text{ in } \omega, \quad u_N = \phi_N \text{ on } \partial\omega.$$

Lemma 2 of [6] shows that

$$\|u_N\|_{1, \omega \setminus D} \leq C N^{-M},$$

while by Lemma 3 of [6]

$$\int_D \rho^\ell |\nabla u_N|^2 d\underline{x} \geq C_\delta N^{-(2+\delta)\ell}, \quad C_\delta > 0,$$

for any $\delta > 0$. A combination of these gives

$$(10) \quad \int_{\omega \setminus D} |\nabla u_N|^2 d\underline{x} \leq C_\delta N^{(2+\delta)\ell - 2M} \int_D \rho^\ell |\nabla u_N|^2 d\underline{x}.$$

By choosing $M > \ell$ and $0 < \delta < \frac{2M}{\ell} - 2$ we obtain $(2+\delta)\ell - 2M < 0$, and hence

$$(11) \quad C_\delta N^{(2+\delta)\ell - 2M} < \varepsilon$$

for sufficiently large N . The corresponding u_N satisfies (i) by definition and satisfies (ii) as a consequence of (10) and (11), using the fact that $\int_D \rho^\ell |\nabla u_N|^2 d\underline{x} \neq 0$.



The next lemma combines the result in Lemma 3 with the Runge Approximation Property to obtain a similar energy estimate for subdomains and with functions that are " γ -harmonic" in the larger domain. For our later application of this result it is essential that the subdomain be only piecewise "smooth". If ω is an analytic curvilinear polygon then we shall call $\underline{x}_0 \in \partial\omega$ a regular boundary point (relative to γ) if there exists a neighborhood B of \underline{x}_0 in \mathbb{R}^2 and a map ϕ such that

- (i) ϕ maps B injectively onto a neighborhood of the origin in \mathbb{R}^2 ,
- (ii) both maps ϕ and ϕ^{-1} have analytic coordinate functions,
- (iii) $\phi(B \cap \omega)$ is given by $\{\underline{x} : x_2 > 0\} \cap \phi(B)$ and γ is analytic on $\overline{B \cap \omega}$.

Lemma 4. Let ω, Ω be two analytic curvilinear polygons with $\omega \subseteq \Omega$.

Let γ be a piecewise analytic function on $\bar{\Omega}$ with $0 < \gamma(\underline{x})$. Assume that $\underline{x}_0 \in \partial\omega$ is regular relative to γ and that \underline{x}_0 lies in the unbounded component of $\mathbb{R}^2 \setminus \omega$. Let D be a neighborhood of \underline{x}_0 relative to $\bar{\omega}$. Given $\ell \geq 0$ and $\epsilon > 0$ there exists $U \in H^1(\Omega)$ such that

$$(i) \quad L_{\gamma} U = 0 \quad \text{in } \Omega$$

$$(ii) \quad \int_{\omega \setminus D} |\nabla U|^2 d\underline{x} < \epsilon \int_D \rho^{\ell} |\nabla U|^2 d\underline{x},$$

where $\rho(\underline{x}) = \text{dist}(\underline{x}, \partial\omega)$.

Proof:

We may without loss of generality assume that $\bar{\omega} \subset \Omega$ and that Ω is simply connected; if not we merely enlarge Ω and extend γ by 1 on this new piece of Ω . Let $\omega' \subseteq \Omega$ be a C^∞ domain so that

- (i) ω' is simply connected and contains ω ,
- (ii) the boundary of ω' coincides with the boundary of ω in a neighborhood of x_0 ,
- (iii) $\overline{\omega \setminus D} \subset \omega'$.

(See Fig. 2.)

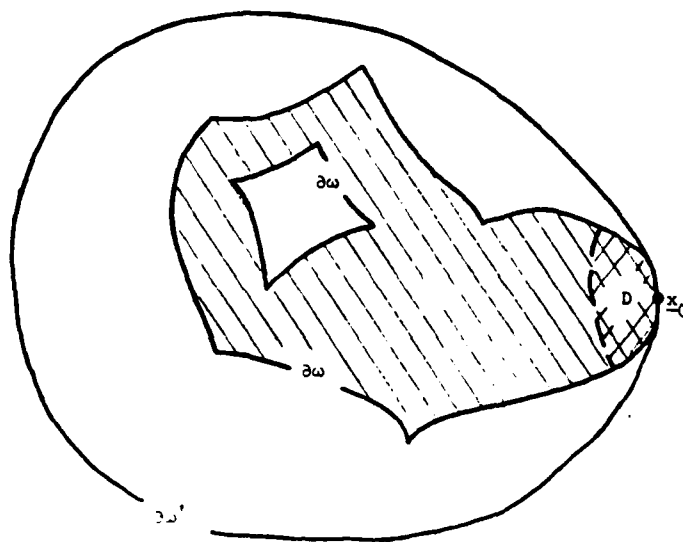


Fig. 2

Choosing a neighborhood $D' \subseteq D$ of x_0 relative to $\bar{\omega'}$, we apply Lemma 3 to construct $u \in H^1(\omega')$ satisfying

$$(12) \quad \int_{\omega' \setminus D'} |\nabla u|^2 d\underline{x} < \delta \int_{D'} (\rho')^\ell |\nabla u|^2 d\underline{x} ,$$

where $\rho'(\underline{x}) = \text{dist}(\underline{x}, \partial\omega')$ and $\delta > 0$ is arbitrary. If D' is small enough we have

$$\rho'(\underline{x}) = \text{dist}(\underline{x}, \partial\omega') = \text{dist}(\underline{x}, \partial\omega) = \rho(\underline{x})$$

for all $\underline{x} \in D'$, and then (12) implies

$$(13) \quad \int_{\omega \setminus D} |\nabla u|^2 d\underline{x} < \delta \int_D \rho^\ell |\nabla u|^2 d\underline{x} .$$

Since the inequality in (13) is strict, there is a compact subset $K_D \subset \text{interior}(D)$ for which

$$(14) \quad \int_{\omega \setminus D} |\nabla u|^2 d\underline{x} < \delta \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} .$$

Let $K = \overline{\omega \setminus D} \cup K_D$. Since it is a compact subset of ω' , Lemma 2 applies to give a function $U \in H^1(\Omega)$ satisfying $L_\gamma U = 0$ in Ω and

$$\int_{\overline{\omega \setminus D} \cup K_D} |\nabla(U-u)|^2 d\underline{x} < \delta \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} .$$

Assuming for simplicity that $\rho \leq 1$ in K_D , we conclude that

$$\int_{\omega \setminus D} |\nabla(U-u)|^2 d\underline{x} + \int_{K_D} \rho^\ell |\nabla(U-u)|^2 d\underline{x} < \delta \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} .$$

It follows using the inequality $\frac{1}{2}|a|^2 \leq |a-b|^2 + |b|^2$ that

$$(15) \quad \left(\frac{1}{2} - \delta\right) \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} < \int_{K_D} \rho^\ell |\nabla U|^2 d\underline{x}$$

and

$$(16) \quad \frac{1}{2} \int_{\omega \setminus D} |\nabla U|^2 d\underline{x} < \delta \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} + \int_{\omega \setminus D} |\nabla u|^2 d\underline{x} .$$

Inequalities (14), (15) and (16) combine to give

$$\begin{aligned} \int_{\omega \setminus D} |\nabla U|^2 d\underline{x} &< 4\delta \int_{K_D} \rho^\ell |\nabla u|^2 d\underline{x} \\ &< \frac{8\delta}{1-2\delta} \int_{K_D} \rho^\ell |\nabla U|^2 d\underline{x} \\ &\leq \frac{8\delta}{1-2\delta} \int_D \rho^\ell |\nabla U|^2 d\underline{x} . \end{aligned}$$

We get the desired inequality by choosing δ so that $8\delta/(1-2\delta) = \epsilon$.



5. Identifiability of piecewise analytic coefficients

Let Ω be an analytic curvilinear polygon. Our goal is to show that two piecewise analytic conductivities γ_1 and γ_2 that produce the same boundary measurements are necessarily equal. It is assumed that $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ with $0 < \gamma_0 \leq \gamma_i(\underline{x})$ ($i=1,2$). "Producing the same boundary measurements" means that the maps

$$\Lambda_{\gamma_i} : \tilde{H}^{1/2}(\partial\Omega) \rightarrow [\tilde{H}^{1/2}(\partial\Omega)]^*$$

defined by

$$\Lambda_{\gamma_i} \phi = \gamma_i \frac{\partial u_i}{\partial \nu} \quad \text{where} \quad L_{\gamma_i} u_i = 0, \quad u_i|_{\partial\Omega} = \phi$$

are the same for $i = 1, 2$. Equivalently, it means that the quadratic forms

$$Q_{\gamma_i}(\phi) = \int_{\Omega} \gamma_i |\nabla u_i|^2 d\underline{x}$$

are the same.

Theorem 1. Let γ_i , $i = 1, 2$, be piecewise analytic functions on $\bar{\Omega}$ with a positive lower bound. If

$$Q_{\gamma_1}(\phi) = Q_{\gamma_2}(\phi)$$

for all $\phi \in \tilde{H}^{1/2}(\partial\Omega)$, then

$$\gamma_1 = \gamma_2.$$

Proof: We shall assume that γ_1 and γ_2 are piecewise analytic relative to the same covering $\{\omega_j\}_{j=1}^N$; by Lemma 1 this represents no loss of generality.

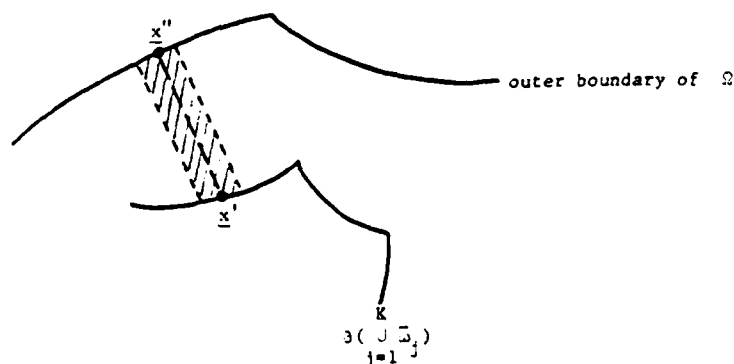
Seeking a contradiction, we suppose that $\gamma_1 \neq \gamma_2$, and we order the cover so that $\{\omega_j\}_{j=1}^K$ are exactly the elements where $\gamma_1 \neq \gamma_2$.

For any analytic curvilinear polygon Ω' , the outer boundary is that part of $\partial\Omega'$ which lies in the unbounded component of $\mathbb{R}^2 \setminus \Omega'$. We claim that there is an analytic curvilinear polygon $\Omega' \subseteq \Omega$ with the following properties:

(17a) Ω' contains all ω_j , $1 \leq j \leq K$

(17b) The outer boundary of Ω' has an analytic curve in common with $\partial\omega_j$ for some j , $1 \leq j \leq K$.

Indeed, if $\partial\omega_j$ has a curve in common with the outer boundary of Ω for some j , $1 \leq j \leq K$, then it suffices to take $\Omega' = \Omega$. Otherwise we choose $\underline{x}' \in \partial(\bigcup_{j=1}^K \overline{\omega_j})$ and \underline{x}'' on the outer boundary of Ω such that \underline{x}' and \underline{x}'' are connected by a piecewise analytic (e.g. piecewise linear) curve lying entirely within $\Omega \setminus \bigcup_{j=1}^K \overline{\omega_j}$, except for the endpoints. We may also assume that $\partial(\bigcup_{j=1}^K \overline{\omega_j})$ is an analytic curve near \underline{x}' , and that $\bigcup_{j=1}^K \omega_j$ lies locally on one side of $\partial(\bigcup_{j=1}^K \overline{\omega_j})$. By excising from Ω a tube along the curve connecting \underline{x}' and \underline{x}'' (with, say, piecewise linear boundaries) we obtain an analytic curvilinear polygon Ω' with the desired properties. (See Fig. 3).



Shaded area marks

"tube" to be cut from Ω .

Fig. 3

For such Ω' , let ω_j , $1 \leq j \leq K$, be a piecewise analytic domain sharing an analytic curve Γ with the outer boundary of Ω' . Since γ_1 and γ_2 are different on ω_j , their Taylor expansions must differ along Γ . Therefore there is a point \underline{x}_0 in the interior of Γ and an integer $\ell \geq 0$ such that

$$|\gamma_1(\underline{x}) - \gamma_2(\underline{x})| \geq c [\text{dist}(\underline{x}, \partial\Omega')]^\ell, \quad c > 0,$$

in an Ω' -neighborhood D of \underline{x}_0 . Note that the boundary point \underline{x}_0 is regular relative to γ_i , $i = 1, 2$, and lies in the unbounded component of $\mathbb{R}^2 \setminus \Omega'$.

Writing $\rho(\underline{x}) = \text{dist}(\underline{x}, \partial\Omega')$, and relabeling if necessary, we have that

$$(18) \quad \gamma_1(\underline{x}) - \gamma_2(\underline{x}) \geq c \rho^\ell(\underline{x}) \quad \text{in } D.$$

Lemma 4 with $\omega = \Omega'$ and $\gamma = \gamma_1$ yields a function $U \in H^1(\Omega)$ such that

$$(19a) \quad L_{\gamma_1} U = 0 \quad \text{in } \Omega$$

$$(19b) \quad \int_{\Omega' \setminus D} |\nabla U|^2 d\underline{x} < \varepsilon \int_D \rho^\ell |\nabla U|^2 d\underline{x},$$

where $\varepsilon > 0$ is a small parameter to be chosen later. Setting

$$\psi = U|_{\partial\Omega} \in \tilde{H}^{1/2}(\partial\Omega),$$

we have

$$(20) \quad \begin{aligned} Q_{\gamma_1}(\psi) &= \int_{\Omega} \gamma_1 |\nabla U|^2 d\underline{x} \\ &= \left(\int_{\Omega \setminus \Omega'} + \int_{\Omega' \setminus D} + \int_D \right) \gamma_1 |\nabla U|^2 d\underline{x}. \end{aligned}$$

Since $\gamma_1 = \gamma_2$ outside Ω' the first integral equals

$$\int_{\Omega \setminus \Omega'} \gamma_2 |\nabla U|^2 d\underline{x} ;$$

the third integral can be bounded from below using (18),

$$\int_D \gamma_1 |\nabla U|^2 d\underline{x} \geq \int_D \gamma_2 |\nabla U|^2 d\underline{x} + c \int_D \rho^\ell |\nabla U|^2 d\underline{x} .$$

Substitution into (20) yields, after deletion of the second integral,

$$\begin{aligned} Q_{\gamma_1}(\psi) &\geq \int_{\Omega \setminus \Omega'} \gamma_2 |\nabla U|^2 d\underline{x} + \int_D \gamma_2 |\nabla U|^2 d\underline{x} + c \int_D \rho^\ell |\nabla U|^2 d\underline{x} \\ (21) \quad &= \int_{\Omega} \gamma_2 |\nabla U|^2 d\underline{x} + c \int_D \rho^\ell |\nabla U|^2 d\underline{x} - \int_{\Omega' \setminus D} \gamma_2 |\nabla U|^2 d\underline{x} . \end{aligned}$$

Applying (19b),

$$\begin{aligned} \int_{\Omega' \setminus D} \gamma_2 |\nabla U|^2 d\underline{x} &\leq (\sup \gamma_2) \int_{\Omega' \setminus D} |\nabla U|^2 d\underline{x} \\ &< \varepsilon \cdot \sup \gamma_2 \int_D \rho^\ell |\nabla U|^2 d\underline{x} . \end{aligned}$$

If $\varepsilon < c/\sup \gamma_2$ ($c > 0$ is the constant in (21)) then it follows that

$$(22) \quad Q_{\gamma_1}(\psi) > \int_{\Omega} \gamma_2 |\nabla U|^2 d\underline{x} .$$

But

$$Q_{\gamma_2}(\psi) = \min_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega} = \psi}} \int_{\Omega} \gamma_2 |\nabla v|^2 d\underline{x} ,$$

so (22) implies that

$$Q_{\gamma_1}(\psi) > Q_{\gamma_2}(\psi) .$$

This contradicts the assumption that γ_1 and γ_2 give the same boundary measurements. Therefore there can be no element ω_j of the cover on which $\gamma_1 \neq \gamma_2$, and the proof is complete.

□

6. Layered structure

In the class of L^∞ (or even C^∞) coefficients γ , identifiability by means of boundary measurements remains an open problem. It is natural to expect, however, that the layered case - which is essentially one dimensional - should be easier. In fact, in this case the Dirichlet- to Neumann-data map determines spectral data for a certain potential, by means of analytic continuation in Fourier space. One dimensional inverse spectral theory is rather well understood, and it will allow us to recover a conductivity γ which is merely three times differentiable. We wish to thank D. Stickler for suggesting the use of analytic continuation to pass from boundary measurements to spectral data.

Throughout this section Ω will be the infinite strip $\{\underline{x} : -\infty < x_1 < \infty, 0 \leq x_2 < 1\} \subseteq \mathbb{R}^2$ and $\gamma = \gamma(x_2)$ will be in $L^\infty((0,1))$ with a positive lower bound. For any pair $(\phi, \psi) \in [H^{1/2}(\mathbb{R})]^2$ there is a unique $u \in H^1(\Omega)$ such that

$$(23) \quad \nabla \cdot (\gamma(x_2) \nabla u) = 0 \text{ in } \Omega, \quad u|_{x_2=0} = \phi, \quad u|_{x_2=1} = \psi.$$

The associated Dirichlet- to Neumann-data map is

$$\Lambda_\gamma(\phi, \psi) = \left(-\gamma \frac{\partial u}{\partial x_2} \Big|_{x_2=0}, \gamma \frac{\partial u}{\partial x_2} \Big|_{x_2=1} \right) \in [H^{-1/2}(\mathbb{R})]^2.$$

We begin by observing that Λ_γ determines γ and $\frac{\partial \gamma}{\partial x_2}$ at $\partial\Omega$ even when γ is merely C^2 .

Lemma 5: If γ_1 and γ_2 are C^2 on $\bar{\Omega}$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then their values and first derivatives agree at $\partial\Omega$.

Proof: We shall check that the argument of [6] applies. First of all, the regularity estimate (7) of that paper holds for $k = 2, 3$ when γ is C^2 .

Therefore formula (14) remains valid for $M = 2$, and the proof of Lemma 3 works

with $t = 3$ in (23). If $\ell \leq 1$ in (26) then the argument on pp. 296-7 applies using $M = 2$, and this yields the desired conclusion. □

Remark 3: This proof does not use the layered structure, but it does use that $\Omega \subset \mathbb{R}^2$. For $\Omega \subset \mathbb{R}^n$ the same argument requires $\gamma \in C^r$, $r > \frac{n}{2}$. Taking into account the layered structure it is possible to reach the conclusion that $\gamma_1 = \gamma_2$ from only knowing that $\Lambda_{\gamma_1}(\phi, \psi) = \Lambda_{\gamma_2}(\phi, \psi)$ for two sets of Dirichlet-data (ϕ, ψ) (see Remark 5). □

Our next task is to separate variables. We shall write $\hat{v}^1 = \hat{v}^1(\xi, x_2)$ for the Fourier transform of a function $v(x_1, x_2)$ with respect to its first argument. (If v only depends on x_1 then we write $\hat{v}(\xi)$; no transforms will ever be taken with respect to x_2 .) From (23), the solution of $L_Y u = 0$ with Dirichlet-data (ϕ, ψ) satisfies

$$(24) \quad \frac{d}{dx_2}(\gamma(x_2) \frac{d}{dx_2} \hat{u}^1(\xi, x_2)) - \xi^2 \gamma(x_2) \hat{u}^1(\xi, x_2) = 0, \quad 0 < x_2 < 1$$

$$\hat{u}^1(\xi, 0) = \hat{\phi}(\xi), \quad \hat{u}^1(\xi, 1) = \hat{\psi}(\xi)$$

for a.e. $\xi \in \mathbb{R}$.

The ODE (24) has its own boundary data map $\Lambda_{\gamma, \xi}$ for each fixed $\xi \in \mathbb{R}$. It takes \mathbb{R}^2 to \mathbb{R}^2 , mapping (α, β) to

$$(25) \quad \Lambda_{\gamma, \xi}(\alpha, \beta) = (-\gamma v'(0), \gamma v'(1)),$$

where $v \in H^1((0, 1))$ solves

$$(26) \quad (\gamma v')' - \xi^2 \gamma v = 0 \quad \text{in } (0, 1), \quad v(0) = \alpha, \quad v(1) = \beta.$$

The two-dimensional boundary data map Λ_Y determines this $\Lambda_{Y,\xi}$ for every ξ :

Lemma 6: If γ_1 and γ_2 are L^∞ with a positive lower bound and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\Lambda_{\gamma_1,\xi} = \Lambda_{\gamma_2,\xi}$ for each $\xi \in \mathbb{R}$.

Proof: Choose Dirichlet-data ϕ and ψ such that $\hat{\phi}$ and $\hat{\psi}$, when restricted to some interval (a,b) , vanish at most on a set of measure zero. From the hypothesis that

$$\Lambda_{\gamma_1}(\phi, 0) = \Lambda_{\gamma_2}(\phi, 0) \quad \text{and} \quad \Lambda_{\gamma_1}(0, \psi) = \Lambda_{\gamma_2}(0, \psi)$$

it follows easily that

$$\Lambda_{\gamma_1,\xi} = \Lambda_{\gamma_2,\xi} \quad \text{for a.e. } \xi \in (a,b) .$$

Since $\Lambda_{Y,\xi}$ is real-analytic as a function of ξ , the same identity must hold for all $\xi \in \mathbb{R}$.



In order to apply inverse spectral theory, we must transform the ODE (26) to a more convenient form. If v solves (26) then $w = \gamma^{\frac{1}{2}}v$ solves

$$(27) \quad -w'' + qw + \xi^2 w = 0 \quad \text{in } (0,1) , \quad w(0) = \gamma^{\frac{1}{2}}(0)\alpha , \quad w(1) = \gamma^{\frac{1}{2}}(1)\beta ,$$

with

$$(28) \quad q = -\frac{1}{4}(\gamma')^2/\gamma^2 + \frac{1}{2}\gamma''/\gamma .$$

We note that

$$w' = \gamma^{\frac{1}{2}}v' + \frac{1}{2}\gamma^{-\frac{1}{2}}\gamma'v .$$

The Dirichlet-to-Neumann map for (27) is

$$\tilde{\Lambda}_{q,\xi} : (w(0), w(1)) \rightarrow (-w'(0), w'(1)) ;$$

clearly it is determined by $\Lambda_{\gamma,\xi}$ and the values of γ and γ' at 0 and 1.

Inverse spectral theory addresses the problem of determining the potential $q(x)$ given knowledge about the eigenfunctions and eigenvalues of $-d^2/dx^2 + q$. The result we shall use is this one, proved in [12]:

Lemma 7. Given two potentials $q_i \in C^1([0,1])$, $i = 1, 2$, consider the operators

$$-\frac{d^2}{dx^2} + q_i$$

with homogeneous Neumann boundary conditions. Let $\lambda_m^{(i)}$ and $\phi_m^{(i)}$ be the associated eigenvalues and eigenfunctions, with the normalization $\phi_m^{(i)}(0) = 1$. If $\lambda_m^{(1)} = \lambda_m^{(2)}$ and $\phi_m^{(1)}(1) = \phi_m^{(2)}(1)$ for each m then $q_1 = q_2$.

It remains to relate the boundary data maps $\tilde{\Lambda}_{q,\xi}$ and the spectral data for q . To this end we consider yet another map $M_{q,\xi}$, taking Cauchy data at 0 to Cauchy data at 1 for the ODE (27). Explicitly,

$$M_{q,\xi}(\alpha, \delta) = (w(1), w'(1)) \quad \text{where}$$

$$-w'' + qw + \xi^2 w = 0 \quad \text{in } (0,1), \quad w(0) = \alpha, \quad w'(0) = \delta.$$

It is easy to check that

$$M_{q_1,\xi} = M_{q_2,\xi} \quad \text{if and only if} \quad \tilde{\Lambda}_{q_1,\xi} = \tilde{\Lambda}_{q_2,\xi}$$

for any q_i of the form (28) and any $\xi \in \mathbb{R}$ (that q_i has the form (28) is

one way to guarantee that $M_{q,\xi}$ is well-defined). For the purpose of identifying γ the use of $M_{q,\xi}$ is thus equivalent to that of $\tilde{M}_{q,\xi}$. An important difference is that $M_{q,\xi}$ is well-defined for complex arguments ξ (and arbitrary $q \in L^\infty$); indeed it extends to a holomorphic function in the entire complex plane. The values of $M_{q,\xi}$ for complex ξ are therefore determined by those for real ξ by analytic continuation. It is easy to show that $M_{q,\xi}$ (with complex ξ) determines the spectral data of q :

Lemma 8: Let q_i , $i=1,2$, be two potentials in $C^1([0,1])$. If $M_{q_1,\xi} = M_{q_2,\xi}$ for all complex ξ , then $q_1 = q_2$.

Proof: Assume that $\lambda^{(1)}$ and $\phi^{(1)}$ is an eigenvalue and the corresponding eigenvector for the operator $-d^2/dx^2 + q_1$ with Neumann boundary conditions,

$$-[\phi^{(1)}]'' + q_1 \phi^{(1)} - \lambda^{(1)} \phi^{(1)} = 0 \quad \text{in } (0,1)$$

$$\phi^{(1)'}(0) = \phi^{(1)'}(1) = 0.$$

As in Lemma 7, we may take the normalization

$$\phi^{(1)}(0) = 1.$$

Choosing $\eta \in \mathbb{C}$ so that $\eta^2 = -\lambda^{(1)}$, and noting that

$$M_{\eta,q_1}(1,0) = M_{\eta,q_2}(1,0),$$

we see that the solution of

$$-w'' + q_2 w - \lambda^{(1)} w = 0, \quad w(0) = 1, \quad w'(0) = 0$$

additionally satisfies

$$w(1) = \phi^{(1)}(1), \quad w'(1) = 0.$$

Therefore $\lambda^{(1)}$ is also an eigenvalue of $-d^2/dx^2 + q_2$, w is the corresponding eigenvector, and its Cauchy data agrees with that of $\phi^{(1)}$ at both endpoints. A similar result holds by symmetry for any eigenvalue and the corresponding eigenvector for q_2 . From Lemma 7 we now conclude that $q_1 = q_2$. \square

We obtain the identifiability of a C^3 , layered conductivity by assembling these results.

Theorem 2: Let Ω be the strip $\{x : -\infty < x_1 < \infty, 0 < x_2 < 1\}$. If $\gamma_1(x_2)$ and $\gamma_2(x_2)$ are two layered conductivities of class $C^3([0,1])$, and if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ (or equivalently if $Q_{\gamma_1} = Q_{\gamma_2}$), then $\gamma_1 = \gamma_2$.

Proof: Let

$$(29) \quad q_i = -\frac{1}{4}(\gamma_i')^2/\gamma_i^2 + \frac{1}{2}\gamma_i''/\gamma_i, \quad i = 1, 2.$$

Lemmas 5 and 6 and the discussion immediately after show that $\tilde{\Lambda}_{q_1, \xi} = \tilde{\Lambda}_{q_2, \xi}$ for all $\xi \in \mathbb{R}$, and we have explained why this implies $M_{q_1, \xi} = M_{q_2, \xi}$ for all $\xi \in \mathbb{C}$. Therefore $q_1 = q_2$ as a consequence of Lemma 8. For fixed q_i (29) is a second-order differential equation for γ_i . Both γ 's have the same Cauchy data, using Lemma 5 again, so they must be equal. \square

Remark 4: One could use other inverse spectral theorems in place of Lemma 7, with minor modifications of the argument. The one proved in [5] requires less regularity for q_i , and it leads to a proof of Theorem 2 with $\gamma_i \in C^2([0,1])$. We believe that the regularity hypotheses on γ_i are an artifact of the method

of proof, and not intrinsic to the problem.



Remark 5: We may conclude that $\gamma_1 = \gamma_2$ based on an (apparent) weaker assumption than $\gamma_1' = \gamma_2'$. It suffices to know that

$$\Lambda_{\gamma_1}(\phi, 0) = \Lambda_{\gamma_2}(\phi, 0) \quad \text{and}$$

$$\Lambda_{\gamma_1}(0, \psi) = \Lambda_{\gamma_2}(0, \psi)$$

for two specific pairs of boundary data $(\phi, 0)$ and $(0, \psi)$ satisfying the condition:

there exists an interval (a, b) , $a < b$,
such that representatives of $\hat{\phi}$ and $\hat{\psi}$,
when restricted to (a, b) , vanish at most
on a set of measure zero.

Indeed, with this assumption, it follows directly from the proof of Lemma 6 that $\Lambda_{\gamma_1, \xi} = \Lambda_{\gamma_2, \xi}$ for each $\xi \in \mathbb{R}$. An energy argument, similar in spirit to that in [6] (or that in the proof of Theorem 1) but applied to the equations

$$(\gamma_1 v')' - \xi^2 \gamma_1 v = 0,$$

with ξ sufficiently large, now shows that $\gamma_1(x) = \gamma_2(x)$ and $\gamma_1'(x) = \gamma_2'(x)$ at $x = 0, 1$. From the discussion following Lemma 6 it follows that $\tilde{\Lambda}_{q_1, \xi} = \tilde{\Lambda}_{q_2, \xi}$, and the rest of the proof of the fact that $\gamma_1 = \gamma_2$ proceeds exactly as before.



7. Convergence of a reconstruction algorithm

The numerical reconstruction of a conductivity γ involves its estimation in some finite dimensional space, using partial information about the boundary measurements. We shall formulate a simple algorithm of this type, and prove its convergence as a consequence of the identifiability results in [6].

If the available data are the energies $\{Q_Y(\phi_m)\}_{m=1}^N$ corresponding to some known boundary values ϕ_1, ϕ_2, \dots , and if γ is to be approximated in a finite dimensional subspace V_N of $L^\infty(\Omega)$, then a natural approach would seem to be

$$\text{find } \gamma_N \in V_N \cap \{\gamma : 0 < \gamma_0 \leq \gamma(x)\} \text{ such that}$$

$$\max_{1 \leq m \leq N} |Q_{\gamma_N}(\phi_m) - Q_Y(\phi_m)| / \|\phi_m\|_{\frac{1}{2}, \partial\Omega}^2 \text{ is "minimal".}$$

For this to succeed, it is of course necessary that γ be approximated well by some element of V_N . That is easily arranged, but it is not enough: since the problem is ill-posed, we must somehow make sure that there is a minimal value and that γ_N can not deviate wildly from γ . A natural way to achieve this is to choose γ_N within a fixed compact set K (relative to a topology in which we expect convergence). Thus modified, our method is to

$$(30) \quad \text{find } \gamma_N \in V_N \cap K \text{ such that}$$

$$\max_{1 \leq m \leq N} |Q_{\gamma_N}(\phi_m) - Q_Y(\phi_m)| / \|\phi_m\|_{\frac{1}{2}, \partial\Omega}^2 \text{ is minimal.}$$

Finding a K which ensures convergence, is not overly conservative, and has a simple description is in general nontrivial. The method (30) could be called semidiscrete, since the evaluation of $Q_{\gamma_N}(\phi_m)$ still requires the

solution of an elliptic boundary value problem. The compactness of K ensures that the sequence $\{\gamma_N\}_{N=1}^{\infty}$ has a subsequence converging to some $\gamma^* \in K$. If the spaces V_N , the set K , and the sequence $\{\phi_m\}_{m=1}^{\infty}$ are chosen properly then γ^* will have the same boundary measurements as γ . If γ is identifiable within K by means of boundary measurements then $\gamma^* = \gamma$ and the method converges.

We give a concrete example, including the details of the convergence proof just sketched, for γ which may be extended holomorphically to a (complex) ball

$$B_R(\underline{x}_0) = \{\underline{z} \in \mathbb{C}^2 : |\underline{z} - \underline{x}_0| < R\}$$

about some $\underline{x}_0 \in \Omega$, with $\bar{\Omega} \subset B_R(\underline{x}_0)$. In other words, we suppose that γ has a power series

$$\gamma(\underline{x}) = \sum_{\alpha} a_{\alpha} (\underline{x} - \underline{x}_0)^{\alpha}, \quad \underline{x} \in \Omega$$

with

$$\sup_{\alpha} R^{|\alpha|} |a_{\alpha}| < \infty, \quad \bar{\Omega} \subset B_R(\underline{x}_0).$$

(In the last two expressions α ranges over all l -tuples of nonnegative integers.)

The functions $\{\phi_m\}_{m=1}^{\infty}$ are assumed to be dense in $H^{\frac{1}{2}}(\partial\Omega)$ (Ω is a C^{∞} domain), and we choose

$$V_N = \{p : p \text{ is a polynomial of degree } \leq d_N\}$$

where $\{d_N\}_{N=1}^{\infty}$ is a sequence converging to ∞ as $N \rightarrow \infty$. The compact set K consists of those analytic functions γ on $\bar{\Omega}$ for which

$$\frac{1}{C} \min_{\underline{x} \in \bar{\Omega}} \gamma(\underline{x}) \leq \gamma \leq C \max_{\underline{x} \in \bar{\Omega}} \gamma(\underline{x})$$

and

$$(31) \quad \sup_{\alpha} \frac{1}{\alpha!} R^{|\alpha|} |D^{\alpha} \zeta(\underline{x}_0)| \leq C \sup_{\alpha} R^{|\alpha|} |a_{\alpha}|$$

for some fixed $C > 1$.

It is easy to see that γ can be approximated well in $V_N \cap K$. Indeed, when N is large enough

$$\zeta_N = \sum_{|\alpha| \leq d_N} a_{\alpha} (\underline{x} - \underline{x}_0)^{\alpha} \in V_N \cap K$$

and this $\zeta_N \rightarrow \gamma$ in $L^{\infty}(\Omega)$ as $N \rightarrow \infty$. We know that

$$(32) \quad |Q_{\zeta}(\phi) - Q_{\gamma}(\phi)| \leq D \|\zeta - \gamma\|_{L^{\infty}(\Omega)} \|\phi\|_{\frac{1}{2}, \partial\Omega}^2$$

whenever ζ and γ are uniformly bounded away from 0 and ∞ . Therefore

$$\min_{\zeta \in V_N \cap K} \max_{1 \leq m \leq N} |Q_{\zeta}(\phi_m) - Q_{\gamma}(\phi_m)| / \|\phi_m\|_{\frac{1}{2}, \partial\Omega}^2 \rightarrow 0$$

as $N \rightarrow \infty$, and so the γ_N satisfy

$$(33) \quad \max_{1 \leq m \leq N} |Q_{\gamma_N}(\phi_m) - Q_{\gamma}(\phi_m)| / \|\phi_m\|_{\frac{1}{2}, \partial\Omega}^2 \rightarrow 0$$

as $N \rightarrow \infty$.

By the definition of K ,

$$\gamma_N = \sum_{\alpha} a_{\alpha}^{(N)} (\underline{x} - \underline{x}_0)^{\alpha}$$

with $a_{\alpha}^{(N)} = 0$ for $|\alpha| > d_N$ and

$$\sup_{\alpha, N} R^{|\alpha|} |a_{\alpha}^{(N)}| \leq C \sup_{\alpha} R^{|\alpha|} |a_{\alpha}|.$$

Therefore there is a subsequence N_k for which

$$a_{\alpha}^{(N_k)} \rightarrow b_{\alpha} \text{ as } k \rightarrow \infty, \text{ for each } \alpha$$

and

$$(34) \quad \sup_{\alpha} R^{|\alpha|} |b_{\alpha}| \leq C \sup_{\alpha} R^{|\alpha|} |a_{\alpha}|.$$

It follows that

$$\gamma_{N_k} \rightarrow \gamma^* = \sum_{\alpha} b_{\alpha} (x-x_0)^{\alpha}$$

uniformly in every ball $\{\underline{x} : |\underline{x}-\underline{x}_0| < R'\}$, $R' < R$, and consequently

$$(35) \quad \gamma_{N_k} \rightarrow \gamma^* \text{ in } L^{\infty}(\Omega).$$

From (33) and (35) we get that

$$Q_{\gamma^*}(\phi_m) = Q_{\gamma}(\phi_m) \quad m=1,2,\dots,$$

and thus

$$(36) \quad Q_{\gamma^*} = Q_{\gamma},$$

since $\{\phi_m\}_{m=1}^{\infty}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$.

The limit γ^* is in \mathcal{S} , using (34), (35) and the fact that $N_k \in K$. From (36) and the identity of analytic conductivities, proved in [6], it follows that $\gamma^* = \gamma$. The preceding

argument shows not only that $\{\gamma_N\}_{N=1}^{\infty}$ has a convergent subsequence tending to γ but also that any convergent subsequence of $\{\gamma_N\}_{N=1}^{\infty}$ necessarily tends to γ . This implies that the entire sequence converges to γ .

Remark 6: The case when γ is merely analytic on $\bar{\Omega}$ can be treated similarly. Let H be a compact set such that

$$\bar{\Omega} \subset \text{interior}(H), \text{ and}$$

γ extends analytically to a neighborhood of H .

The latter condition assures that

$$\sup \frac{1}{\alpha!} R_H^{|\alpha|} |D^\alpha \gamma(\underline{x})| < \infty,$$

for some $R_H > 0$, where the supremum is now over points $\underline{x} \in H$ and integer two-tuples $\alpha \geq 0$. If (31) is replaced by

$$\sup \frac{1}{\alpha!} R_H^{|\alpha|} |D^\alpha \zeta(\underline{x})| \leq C \sup \frac{1}{\alpha!} R_H^{|\alpha|} |D^\alpha \gamma(\underline{x})|$$

in the definition of K (the supremum again being over $\underline{x} \in H$ and $\alpha \geq 0$) then the argument can proceed essentially as before.

Remark 7: A similar semidiscrete method can be formulated for the layered case. By Theorem 2, convergence can be assured by choosing K so that four derivatives of γ remain uniformly bounded on $\bar{\Omega}$.

Remark 8: For piecewise analytic γ we know identifiability from Theorem 1, but it is not so clear now to define K . If a cover relative to which K is piecewise analytic is known, then one can proceed as before by controlling γ_N separately on each piece. If the cover is unknown, however, it seems difficult to find a choice of K which assures convergence.

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